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Holographic Phase Transition to Topological Dyons

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Abstract

The dynamical stability of a Julia-Zee solution in the AdS background in a four dimensional Einstein-Yang-Mills-Higgs theory is studied. We find that the model with a vanishing scalar field develops a non-zero value for the field at a certain critical temperature which corresponds to a topological dyon in the bulk and a topological phase transition at the boundary.

1 Introduction

The gravity-gauge theory duality, AdS/CFT [1] [2] [3], has provided the hope of doing quantitative analysis in low energy QCD. It is suspected that the phenomenon of confinement in QCD may be related to the condensation of topological configurations, specifically magnetic monopoles in QCD [4] [5] [6] . Extended topological configurations in field theories have other significant consequences.

In this work we study the consequence of one such topological charged magnetic configuration, a dyon, in the bulk of an AdS/ CFT set up. The configuration being the dyon solution of an SU(2) gauge theory coupled to a triplet of scalar fields, of 'tHooft-Polyakov-Julia-Zee type, in the asymptotically AdS Reissner-Nordstrom blackhole background. We will argue that this will emerge in the low temperature regime from a configuration with no scalar field. The boundary theory will make a corresponding phase transition.

Magnetic monopoles have a long history. The gauge theory in 4 dimensions, the electrodynamics, does not allow magnetic charges at the right hand side of the Maxwell equation for the dual field intensity vanishes identically, unless a singularity, the Dirac string, is inserted. A similar situation in non-abelian gauge theories persists.

Nearly thirty five years ago Polyakov and 'tHooft observed that introduction of an adjoint scalar permits the appearance of a magnetic monopole solution in the SU(2) gauge theory, upon spontaneous breaking of the gauge symmetry to a U(1) subgroup determined by the direction of the scalar field in isospin space. The magnetic charge being a topological quantity, is classically quantized [7] [8].

This "Hedge-hog" solution led to an avalanche of work on generalization to other gauge groups with serious problems for cosmology, whose resolution was achieved in the theory of inflation.

Shortly after the discovery of the solutions with magnetic charge, dyons with electric charge also were found by Julia and Zee [9]. A particular limit in the solutions discovered by Prasad and Sommerfield [10], saturating a bound observed by Bogomol'nyi [11],the BPS solution has permeated the literature of supergravity and string theory.

Magnetic monopole solutions in the presence of gravity have also attracted a great deal of attention. Many solutions in asymptotically flat, deSitter, or Anti-deSitter spaces with and without blackhole singularity have been discovered [12]. Remarkably only the ones with the Anti-deSitter asymptotics are stable [13] [14]. This is a striking result in view of the AdS/CFT duality.

In particular for SU(2) gauge theory, asymptotically AdS blackhole solutions with and without scalar field have been discovered [15], and more recently by Lugo, Moreno, and Schaposnik, [16].

We will use a particularly simple solution found quite some time ago by Kasuya and Kamata [17], in the present work. We will find that their solution in the absence of a scalar field is unstable while the one with non-vanishing scalar triplet is.

The paper is organized as follows. In the next section we briefly review the Julia-Zee dyon in flat space. In section 3 we introduce it's extension to gravitational versions. In section 4 we review the general procedure for studying dynamical stability. In section 5 we apply this procedure to our case and derive the main analytic equations. In section 6 we present the results of our numerical calculations. Section 7 is devoted to conclusion.

When this work was completed, a paper by Lugo, Moreno, and Schaposnik appeared [18], with similar considerations, related to their previous solution and in the context of the noncompact boundary of R^2 in place of S^2.

2 Julia Zee dyon in flat space

In 1974 'tHooft [7] and Polyakov [8] independently introduced a new type of magnetic monopole in a flat Minkowski space. Their monopole is free of any singularities. The action of the theory is Yang-Mills-Higgs with a special nontrivial ansatz for the gauge and scalar fields. The action is:

$$S = \int d^4x \, [-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi^a\phi^a)], \tag{2.1}$$

The scalar potential is assumed to have a minimum at $\phi^a\phi^a = (\vec{\phi})^2 = const \neq 0$. The covariant derivative and the field strength are defined as usual,

$$D_\mu\phi^a = \partial_\mu\phi^a + e\epsilon^{abc}A^b_\mu\phi^c \; ; \; F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu. \tag{2.2}$$

The 'tHooft-Polyakov ansatz for a solution for this system is,

$$\vec{\phi} = \frac{H(r)}{er}\hat{r} \; ; \; \vec{A}_i = \frac{1-k(r)}{er}\vec{a}_i \; (i=1,2,3) \; ; \; a^n_i = \epsilon^{nik}\hat{x}^k \; ; \vec{A}_t = 0 \tag{2.3}$$

Regularity and finiteness of energy requires the boundary conditions,

at $r \to 0$: $H \to 0 \, ; K \to 1$
at $r \to \infty$: $H \to const \times r$; $K \to 0$

The U(1) field strength, invariant under the su(2) algebra, is defined as,

$$F_{\mu\nu} = \vec{F}_{\mu\nu}.\hat{\phi} + \hat{\phi}.[D_\mu\hat{\phi} \times D_\nu\hat{\phi}] \; ; \; \hat{\phi} = \frac{\vec{\phi}}{\sqrt{\vec{\phi}.\vec{\phi}}} \; ; \tag{2.4}$$

When $r \to \infty$, the magnetic field scales as $1/r^2$; so the configuration has a magnetic charge:

$$\lim_{r\to\infty} B_i \to \frac{1}{e\,r^2} \Rightarrow \oint B_i \, ds^i = \frac{1}{e} = g \tag{2.5}$$

Later Arafune, Freund, and Goebel [19] showed that this magnetic charge is a topological object and its value is discrete,

$$g = \oint B_i \, ds^i = \frac{n}{e}. \tag{2.6}$$

The hamiltonian of the system can be written as,

$$H = \int d^3x \, [\frac{1}{4}\big(F^a_{ij} - \epsilon_{ijk}D_k\phi^a\big)^2 + V(\phi^a\phi^a)] + 4\pi g\langle\phi\rangle, \tag{2.7}$$

where $\langle\phi\rangle$ is the boundary value of ϕ which is a constant by assumption. It is then easily found [11] that there is a bound on the energy, BPS bound,

$$E \geq 4\pi g\langle\phi\rangle. \tag{2.8}$$

The bound is saturated when

$$F^a_{ij} = \epsilon_{ijk} D_k \phi^a. \tag{2.9}$$

This magnetic monopole solution was later extended to a dyon by Julia and Zee [9]. Their solution was in the form:

$$\begin{aligned}
\vec{\phi} &= \frac{H(r)}{er}\hat{r}, \\
\vec{A}_\alpha &= \frac{1-K(r)}{e}\vec{a}_\alpha, \\
\vec{A}_t &= \frac{J(r)}{er}\hat{r}.
\end{aligned} \tag{2.10}$$

The boundary conditions for H and K are as before, and the boundary conditions for $J(r)$ are:

at $r \to 0$: $J \to 0$
at $r \to \infty$: $J \to const \times r$

With this ansatz we see that the configuration has an electric charge too.

In 1975 Prasad and Sommerfield [10] found an exact analytic solution for this dyon in the limit $V \to 0$. their solution has the form:

$$\begin{aligned}
K &= \frac{Cr}{sinh(Cr)}, \\
J &= sinh(\gamma)\big[Cr\ coth(Cr) - 1\big], \\
H &= cosh(\gamma)\big[Cr\ coth(Cr) - 1\big],
\end{aligned} \tag{2.11}$$

where C and γ are arbitrary constants.

3 Julia-Zee dyon coupled to gravity

The dyon of the previous section lives in flat space but what happens if we include the gravity? During 80's it was shown that there exist gravitational Julia-Zee dyons; and certain exact solutions were obtained [20]. One of the simplest solutions in this class is the Kasuya and Kamata solution [17].

When gravity is included the action is:

$$S \ = \int \sqrt{-g}\ d^4x\ \big[\frac{1}{16\pi G}(R + \frac{6}{L^2}) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D_\mu\phi^a D^\mu\phi^a\ - \lambda\ V(\phi^a\phi^a)\big] \tag{3.1}$$

And as before the scalar potential is assumed to have a minimum at some nonzero constant $(\vec{\phi})^2$. The equations of motion are:

$$
\begin{aligned}
& R_{\mu\nu} - \frac{1}{2}(R + \frac{6}{L^2})\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \\
& T_{\mu\nu} = \big[\frac{\Lambda}{16\pi G} - \frac{1}{4}F^a_{\rho\lambda}F^{a\rho\lambda} - \frac{1}{2}D_\lambda\phi^a D^\lambda\phi^a - \lambda V(\phi^2)\big] g_{\mu\nu} + F^a_{\mu\lambda}F^{a\lambda}_{\nu} + D_\mu\phi^a D_\nu\phi^a, \\
& \\
& \partial_\mu\big(\sqrt{-g}D^\mu\phi^a\big) + \sqrt{-g}\big(e\epsilon^{abc}A^b_\mu D^\mu\phi^c - \lambda\,\frac{\delta V}{\delta\phi^a}\big) = 0, \\
& \\
& \partial_\mu\big(\sqrt{-g}F^{a\mu\nu}\big) - e\sqrt{-g}\;\epsilon^{abc}\big(F^{b\mu\nu}A^c_\mu + (\partial^\mu\phi^b)\;\phi^c\big) + e^2\sqrt{-g}\big((A^{b\nu}\phi^b)\phi^a - (\phi^b\phi^b)A^{a\nu}\big) = 0.
\end{aligned}
\tag{3.2}
$$

The same ansatz for scalar and vector fields is assumed as in the flat space case. The most general form for a spherically symmetric metric in 4 dimensions is:

$$
g_{\mu\nu} \equiv diag(-e^{X(r)}, e^{Y(r)}, r^2, r^2 sin(\theta)^2) \tag{3.3}
$$

If the scalar potential is taken to be in the form of a mexican hat, $V = \frac{1}{4}(\phi^a\phi^a)^2 - \frac{v^2}{2}(\phi^a\phi^a)$, then the equtions of motion for H, K and J become:

$$
\begin{aligned}
& J'' - \frac{r}{2}(X+Y)'\big(\frac{J}{r}\big)' - e^Y\frac{2JK^2}{r^2} = 0, \\
& K'' + \frac{1}{2}(X-Y)'K' - e^Y\frac{K}{r^2}\big(K^2 + H^2 - e^{-X}J^2 - 1\big) = 0, \\
& H'' + \frac{r}{2}(X-Y)'\big(\frac{H}{r}\big)' - e^Y\frac{H}{r^2}\big(2K^2 + \frac{\lambda}{e^2}(H^2 - c^2r^2)\big) = 0,
\end{aligned}
\tag{3.4}
$$

where $c^2 = e^2v^2$ and v is the minimum of the scalar potential.
The equations of motion for the metric yield,

$$
\begin{aligned}
& \frac{e^{-Y}}{r^2}(rY' - 1) + \frac{1}{r^2} = \frac{8\pi G}{e^2}T^t_t, \\
& -\frac{e^{-Y}}{r^2}(rX' + 1) + \frac{1}{r^2} = \frac{8\pi G}{e^2}T^r_r, \\
& -\frac{e^{-Y}}{2}\big[X'' + \frac{1}{2}(X')^2 - \frac{1}{2}X'Y' + \frac{1}{r}(X-Y)'\big] = \frac{8\pi G}{e^2}T^\theta_\theta = \frac{8\pi G}{e^2}T^\varphi_\varphi,
\end{aligned}
\tag{3.5}
$$

where $T^t_t, T^r_r, T^\theta_\theta$ and T^φ_φ are the components of the energy momentum tensor,

$$\begin{aligned}
T^t_t &= \Big[\frac{e^{-Y}}{r^2}(K')^2 + \frac{(K^2-1)^2}{2r^4} + \frac{e^{-(X+Y)}}{2}[(\frac{J}{r})']^2 + \\
&e^{-X}\frac{J^2K^2}{r^4} + \frac{e^{-Y}}{2}[(\frac{H}{r})']^2 + \frac{H^2K^2}{r^4} + e^2\lambda V(\phi)\Big], \\
T^r_r &= \Big[-\frac{e^{-Y}}{r^2}(K')^2 + \frac{(K^2-1)^2}{2r^4} + \frac{e^{-(X+Y)}}{2}[(\frac{J}{r})']^2 - \\
&e^{-X}\frac{J^2K^2}{r^4} - \frac{e^{-Y}}{2}[(\frac{H}{r})']^2 + \frac{H^2K^2}{r^4} + e^2\lambda V(\phi)\Big], \\
T^\theta_\theta &= T^\varphi_\varphi = \Big[-\frac{(K^2-1)^2}{2r^4} - \frac{e^{-(X+Y)}}{2}[(\frac{J}{r})']^2 + \frac{e^{-Y}}{2}[(\frac{H}{r})']^2 + e^2\lambda V(\phi)\Big].
\end{aligned} \tag{3.6}$$

Kasuya and Kamata found an exact solution to these equations with the above ansatz [17]:

$$H = c\,r\ ;\ K = 0\ ;\ J = \mu\, r - \rho\ ;\ e^X = e^{-Y} = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{r^2}{L^2}, \tag{3.7}$$

where,

$$q^2 = \frac{4\pi G_N(1+\rho^2)}{e^2}. \tag{3.8}$$

This expression is still a solution if we set $H = 0$. The background metric in both cases are *AdS-RN*; but, there is an important point : The AdS radius differs for the two, because the contribution of the scalar potential to the cosmological constant is different for the two cases. Note also that because the gauge field should have a finite norm, A^a_t should be zero at the horizon of the blackhole, implying a relation between μ and ρ

$$\mu = \frac{\rho}{r_H}, \tag{3.9}$$

where r_H is the outer horizon of the blackhole. The temperature of the blackhole after eliminating the charge of the blackhole from definition of the horizon ($e^X = 0$) is:

$$T = \frac{1}{2\pi r_H}\big(1 - \frac{M}{r_H} + \frac{2r_H^2}{L^2}\big). \tag{3.10}$$

Concerning the Bogomol'nyi equation, when the gravity is included, the simplest guess is:

$$D_i\phi^a = \frac{1}{2}\sqrt{-g}\ \epsilon_{ijk}F^{ajk}. \tag{3.11}$$

But this does not work; in fact this relation is not compatible with the equations of motion derived from the action (3.1). It has been shown that the action of the theory can be changed in such a way that its equations of motion are compatible with the relation (3.11) [21]. On the other hand in certain circumstances a Bogomol'nyi like equation can be written which

is compatible with the action (3.1). For example one may consider a generalization of the Bogomol'nyi equation (2.9) of the form,

$$D_i\phi^a \;+\; \partial_i(u)\,\phi^a = \frac{1}{2}\sqrt{-g}\;\epsilon_{ijk}F^{ajk}, \tag{3.12}$$

where u is an additional function. Consistency of this equation with the equations of motion leads to:

$$u = log(\sqrt{|g_{00}|}\;)\;;\;\Delta u = 0, \tag{3.13}$$

where Δ is the usual covariant Laplacian; this in fact is a constraint on the metric. For more details see [22].

4 General aspects of dynamical stability

In this section we review the general aspects of dynamical stability analysis and holographic phase transition.

In the presence of a background solution for a system, checking dynamical stability is to study the time evolution of the solution. Consider a system with a set of fields $X_i(x)$ and the equations of motion $E_i[X_j(x)] = 0$, and a set of exact solutions $X_i^{(0)}(x)$. Varying the solution with an infinitesimal time dependent variation, the linearized equation will be,

$$L_i[X_j^{(0)}, \delta\tilde{X}_j(x), \omega] = 0, \tag{4.1}$$

with $\delta X_i(x,t) = \delta\tilde{X}_i(x)\;e^{-i\omega t}$.

If these equations admit a solution with an ω which has positive imaginary part, then the variations blow up with time and the background solution will be unstable.

This simple argument is the core of the concept of dynamical instability, but there are a number of technical issues concerning the boundary conditions on the fields which are important:

The variations should be regular at any point of the space, as we assumed these variations to be infinitesimal.

Also, at the horizon because of the general properties of the non-extremal blackholes, the radial dependence of the variations near the horizon is generally of the form [23],

$$\delta\tilde{X}_i(r) = (r - r_H)^{\pm i\frac{\omega}{4\pi T}}\big(a_0 + a_1(r - r_H) + a_2(r - r_H)^2 + ...\big), \tag{4.2}$$

where T is the temperature of the blackhole, and the $\pm$ signs indicate the ingoing and outcoming modes. Then as a classical blackhole devours everything, the solution near the horizon should be ingoing only.

Moreover, at the boundary at infinity, because the equations are of order two, the solution asymptotically has the form,

$$\delta\tilde{X} \sim\; c_1\, r^{\Delta_+} \;+\; c_2\, r^{\Delta_-}\;. \tag{4.3}$$

There are then two possibilities:

1) One of the modes diverges (non-normalizable); the diverging mode must be excluded as the variation should remain small; while the other one vanishes at the boundary (normalizable) .

2) Both modes vanish. Then the one that is coupled to the appropriate boundary operator should be chosen. In the context of AdS/CFT such normalizable modes are interpreted as the v.e.v of the corresponding operator at the field theory (CFT) side.

So our boundary conditions are:

$$\begin{aligned} &\text{near horizon} && \rightarrow \text{ingoing mode,} \\ &\text{at the boundary} && \rightarrow\ c_1\, r^{\Delta_+}\ +\ c_2\, r^{\Delta_-}. \end{aligned} \tag{4.4}$$

The coefficients c_1 and c_2 are functions of parameters in the theory, e.g. temperature, radius of the blackhole, and chemical potential. Then choosing one of the boundary modes, e.g. $c_1 = 0$, leads to the allowed frequencies:

$$c_1(\,T, \omega, r_H, \mu, ...) = 0\ \rightarrow\ \omega(\,T, r_H, \mu, ...). \tag{4.5}$$

Then if the imaginary part of the ω changes sign in the vicinity of a hypersurface (*wall of marginal stability*) in the $(\,T, r_H, \mu, ...)$ space, then there will be a *stability/instability* transition for the configuration, and the system goes through a phase transition.

The modes on the wall of marginal stability with ω equal to zero are called *marginally stable modes* [24].

5 Stability analysis of gravitational Julia-Zee like solution

We consider the quartic potential:

$$\lambda V(\phi^a\phi^a) = \frac{\lambda}{4}(\phi^a\phi^a)^2 - \frac{1}{L^2}(\phi^a\phi^a). \tag{5.1}$$

As mentioned in section 3, there are two exact solutions to the equations of motion for Julia-Zee ansatz in the form of (2.10): $H = 0$ with AdS radius L; and $H =\ r\,\sqrt{\frac{2e^2}{\lambda L^2}}$ with AdS radius $\tilde{L}$:

$$\frac{1}{\tilde{L}^2} = \frac{1}{L^2} + \frac{8\pi G_N}{3\lambda L^4}. \tag{5.2}$$

Perturbing the matter fields,

$$\begin{aligned} \frac{H}{er} &\rightarrow \frac{H}{er} + \epsilon\, e^{-i\omega\, t}\ \frac{f(r)}{e}, \\ \frac{J}{er} &\rightarrow \frac{J}{er} + \epsilon\, e^{-i\omega\, t}\ \frac{P(r)}{e}, \\ \frac{K}{e} &\rightarrow \frac{K}{e} + \epsilon\, e^{-i\omega\, t}\ \frac{Q(r)}{e}, \end{aligned} \tag{5.3}$$

where ϵ is an infinitesimal parameter, and putting the new fields in the equations of motion for the solution $H = 0$, to first order in ϵ, we get:

$$
\begin{array}{lll}
\phi & equation: & \big(r^2 e^X\, f'\big)' + \big(\frac{2}{L^2} + e^{-X}\omega^2\big)\, r^2\, f = 0, \\
A_t & equation: & 2P' + r\, P'' = 0, \\
A_r & equation: & P' = 0, \\
A_\theta & equation: & \begin{cases} Q'' + X'Q' + e^{-2X}\big[e^X/r^2 + (\mu - \rho/r)^2 + \omega^2\big]Q = 0, \\ Q = 0, \end{cases} \\
A_\varphi & equation: & \begin{cases} Q'' + X'Q' + e^{-2X}\big[e^X/r^2 + (\mu - \rho/r)^2 + \omega^2\big]Q = 0, \\ Q = 0. \end{cases}
\end{array}
\tag{5.4}
$$

The solutions to the gauge field perturbations are simply $P = const$, $Q = 0$; but, the equation for the variation of ϕ is nontrivial. We can consider different perturbations to A_φ and A_θ, but the A_t equation implies that they should be equal.

The first step is to determine the boundary behavior of the function f. The near horizon equation for f is:

$$
f'' + \frac{f'}{u} + \big(\frac{\tilde{\eta}}{u} + \frac{\tilde{\omega}^2}{u^2}\big) f = 0, \tag{5.5}
$$

where u is $(r - r_H)$, and,

$$
\tilde{\eta} = \frac{1}{2\pi T L^2}\ ;\ \tilde{\omega} = \frac{\omega}{4\pi T}, \tag{5.6}
$$

with the solution,

$$
J_{\pm 2i\sqrt{\tilde{\omega}^2}}(2\sqrt{\tilde{\eta}\, u}), \tag{5.7}
$$

which has the small u expansion,

$$
u^{\pm i\sqrt{\tilde{\omega}^2}}\ \big(\frac{\tilde{\eta}^{\pm i\sqrt{\tilde{\omega}^2}}}{\Gamma[1 \pm 2i\sqrt{\tilde{\omega}^2}]} - \frac{\tilde{\eta}^{1\pm i\sqrt{\tilde{\omega}^2}}}{\Gamma[2 \pm 2i\sqrt{\tilde{\omega}^2}]}\, u + \frac{\tilde{\eta}^{2\pm i\sqrt{\tilde{\omega}^2}}}{2\Gamma[3 \pm 2i\sqrt{\tilde{\omega}^2}]}\, u^2 + O[u]^3 \big). \tag{5.8}
$$

This corresponds to the form (4.2).

The boundary equation for f at $r \to \infty$ is :

$$
f'' + \frac{4}{r} f' + \big(\frac{2}{r^2} + \frac{L^4\omega^2}{r^4}\big) f = 0. \tag{5.9}
$$

Changing the variable to $z = 1/r$,

$$
f'' - \frac{2}{z} f' + \big(\frac{2}{z^2} + L^4\omega^2\big) f = 0, \tag{5.10}
$$

which has the solution,

$$
f = z\, e^{\pm i L^2\omega\, z} = \frac{e^{\pm i \frac{L^2\omega}{r}}}{r}. \tag{5.11}
$$

For large r this has the form,

$$
f \sim \frac{f_1}{r} + \frac{f_2}{r^2} + ... \tag{5.12}
$$

As described in the last section a choice has to be made; at the horizon the ingoing mode is to be chosen and at the boundary at infinity either f_1 or f_2 should be set equal to zero. Each leads to a different value for the critical temperature.

We can repeat exactly the same procedure for the second solution with $H = \ r\sqrt{\frac{2e^2}{\lambda L^2}}$. At the linear order the equations are as before except for the ϕ equation,

$$\big(r^2 e^X\ f'\big)' + \big(-\frac{4}{L^2} + e^{-X}\omega^2\big)\ r^2\ f = 0. \tag{5.13}$$

The forms of this equation near the horizon and the boundary at infinity are:

$$\begin{aligned} f'' + \frac{f'}{u} + \big(-\frac{\tilde{\eta}}{u} + \frac{\tilde{\omega}^2}{u^2}\big) f = 0, \\ f'' + \frac{4}{r} f' + \big(-\frac{4\alpha^2}{r^2} + \frac{(\alpha L)^4\omega^2}{r^4}\big) f = 0, \end{aligned} \tag{5.14}$$

where

$$\tilde{\eta} = \frac{1}{\pi T \tilde{L}^2}\ ,\ \tilde{\omega} = \frac{\omega}{4\pi T}\ ,\ \alpha = \frac{\tilde{L}}{L} = \frac{1}{\sqrt{1+\frac{\gamma}{L^2}}}\ ,\ 0 \le \alpha \le 1, \tag{5.15}$$

γ is $(8\pi G_N)/(3\lambda)$, $u = r - r_h$, and " $'$ " in the first equation denotes derivation with respect to u and in the second equation with respect to r.

The solutions to the near horizon equation are:

$$I_{\pm 2i\sqrt{\tilde{\omega}^2}}(2\sqrt{u\tilde{\eta}}), \tag{5.16}$$

which are ingoing and outcoming modes. The solutions to the equation at the boundary at infinity are:

$$\frac{1}{r^{3/2}} J_{\pm\frac{1}{2}\sqrt{9+16\alpha^2}}\big(\frac{L\alpha^2\omega}{r}\big), \tag{5.17}$$

with the asymptotic expansion of the form,

$$\begin{aligned} f \ &= \ f_1\, r^{\Delta_+} \ + \ f_2\, r^{\Delta_-}, \\ \Delta_\pm &= \frac{-3 \pm \sqrt{9+16\alpha^2}}{2}. \end{aligned} \tag{5.18}$$

The first term is divergent and is not acceptable, thus we set $f_1 = 0$.

6 Numerics

We follow the procedure outlined above for finding the *phase transition* temperature by setting ω equal to zero, i.e., finding the marginally stable mode.

6.1 The case $\phi = 0$:

6.1.1 marginally stable modes:

The ϕ equation when ω is set to zero is:

$$\left(r^2 e^X\ f'\right)' + \frac{2}{L^2}\, r^2\, f = 0. \tag{6.1}$$

The forms of this equation at the horizon and at the boundary at infinity are:

$$\begin{aligned} f'' + \frac{f'}{u} + \frac{\tilde{\eta}}{u} f = 0, \\ f'' + \frac{4}{r} f' + \frac{2}{r^2} f = 0, \end{aligned} \tag{6.2}$$

where $u = r - r_H$; and " $'$ " denotes derivation with respect to u in the first equation and to r in the second one.

The solutions to the first equation are:

$$J_0(2\sqrt{\tilde{\eta}\, u})\ ;\ Y_0(2\sqrt{\tilde{\eta}\, u}). \tag{6.3}$$

Now $Y_0(2\sqrt{\tilde{\eta}\, u})$ is divergent at the horizon so it is ruled out; thus we choose $J_0(2\sqrt{\tilde{\eta}\, u})$, and its near horizon expansion is:

$$1 - \tilde{\eta}\, u\ +\ .... \tag{6.4}$$

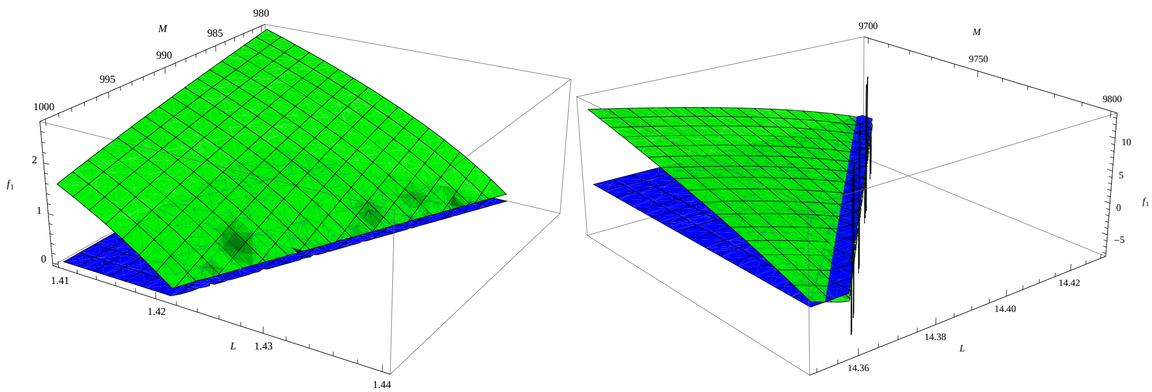
We can fix f at the horizon to be equal to 1.

The solution to the second equation is,

$$f = \frac{f_1}{r} + \frac{f_2}{r^2}. \tag{6.5}$$

Eliminating the charge of the blackhole in favor of M, L and r_H, then f_1 and f_2 will be functions of (M, L, r_H). Imposing the desired boundary condition, $f_1 = 0$ or $f_2 = 0$, we will find a relation between (M, L, r_H). Simultaneously we have the positivity condition of the temperature , $T(M, L, r_H) \geq 0$. For convenience we fix r_H to a certain value, e.g. $r_H = 10$, compute f_1 or f_2 numerically and plot it as a function of M and L (by considering the positivity condition of blackhole temperature, $-ML^2 + 2r_H^3 + r_H L^2 \geq 0$). Then we will have a surface, in the space of (M, L, f_1) or (M, L, f_2) . Where this surface intersects with the plane $f_1 = 0$ or $f_2 = 0$, is the critical curve of (M_c, L_c). If there is no intersection then there is no marginally stable mode for the considered value of r_H with the desired boundary condition at infinity. Our calculations show that for *sufficiently large radius of the blackhole* such an intersection exists.

In figure (1) we have shown focused plots of f_1 for two values of r_H. For $r_H = 10$ there is no intersection, but, for $r_H = 100$ we have an intersection.



(a) (b)

Figure 1: The green surfaces are the focused plots of f_1 as a function of M and L and the blue planes are the planes for which f_1 vanishes (plot (a) for $r_H = 10$ and plot (b) for $r_H = 100$). The plots are cut by $T \geq 0$ condition.

It can be seen that f_1 and f_2 can vanish several times before the temperature becomes zero.

6.1.2 Quasi-normal modes:

In this section we introduce our numerical results for quasi-normal modes. Our aim is to solve the first equation of (5.4) with the desired boundary conditions and find the complex frequencies. Our numerical calculations show that at high temperatures the $\phi = 0$ background is stable, the imaginary part of the quasi-normal frequencies are negative; and at sufficiently low temperatures this background is unstable, the imaginary part of the quasi-normal frequencies are positive.

It is obvious that we can not sweep the entire complex frequency plane by numerical calculations but it seems that at sufficiently high temperatures there is only one stable mode for every given temperature (or at least the frequencies in complex frequency plane are so widely distributed that we could only find one of them); but, at lower temperatures there are a number of unstable modes for every value of temperature. The quasi-normal frequencies that we found are listed in the tables below for both $f_1 = 0$ and $f_2 = 0$.

Note that, specially for $f_1 = 0$, in a range of temperatures above the onset of instability, the quasi-normal frequencies are purely imaginary or their real part is very small; this may be the sign of the existence of a mass gap right above the critical temperature T_c.

Note also that for $f_1 = 0$, although there is no marginally stable mode at $r_H = 10$, as mentioned in previous subsection, the quasi-normal frequencies exhibit a phase transition. This means that for the boundary conditions under consideration, when the temperature is changed, the frequencies do not go through the origin of the complex frequency plane.

Table 1: quasi-normal modes for $f_1 = 0$

$r_H = 10$, $L = 1$		
M	4π T	ω
2000	0.212	0.764969 + 1.66858 i
2000	0.212	4.93743 + 3.35654 i
1970	0.812	10.133 + 36.3126 i
1970	0.812	0.669683 + 9.05345 i
1950	1.212	9.9903 + 34.2773 i
1900	2.211	0.0148391 + 35.9313 i
1850	3.211	12.5418 + 52.0683 i
1800	4.210	4.98854×10^{-15} - 1.30046 i
1700	6.210	5.26857×10^{-14} - 2.09467 i
1600	8.209	2.36149×10^{-14} - 3.01488 i
1500	10.21	1.26155×10^{-9} - 4.13518 i
1200	16.21	2.16507 - 7.78775 i
1200	16.21	8.66376 - 7.8904 i
1000	20.20	3.7928 - 9.49205 i

Table 2: quasi-normal modes for $f_2 = 0$

$r_H = 10$, $L = 1$		
M	4π T	ω
2000	0.212	1.06157 + 1.95297 i
1990	0.412	1.09226 + 3.81241 i
1980	0.612	1.0839 + 5.7522 i
1950	1.212	3.31773 + 19.6673 i
1900	2.211	1.77525×10^{-6} + 0.320354 i
1500	10.21	0.0000327886 - 0.968159 i
1400	12.207	0.0000337517 - 1.45497 i
1000	20.20	1.55273 - 5.85023 i
800	24.20	4.1098 - 5.3992 i

6.2 The case $\phi = \sqrt{\frac{2}{\lambda L^2}}$:

We have also examined marginally stable modes for The Kasuya-Kamata solution (3.7). In this case the equations for f when ω is set to zero, are,

$$\begin{aligned} &\textit{full equation} : && \left(r^2 e^X f'\right)' - \frac{4}{L^2} r^2 f = 0, \\ &\textit{near horizon equation} : && f'' + \frac{f'}{u} - \frac{\tilde{\eta}}{u} f = 0, \\ &\textit{boundary at } \infty \textit{ equation} : && f'' + \frac{4}{r} f' - \frac{4\alpha^2}{r^2} f = 0. \end{aligned} \tag{6.6}$$

The solutions to the near horizon equation are:

$$I_0(2\sqrt{\tilde{\eta}\, u})\ ;\ K_0(2\sqrt{\tilde{\eta}\, u}). \tag{6.7}$$

The second solution is divergent at the horizon so we choose the first one, which has the expansion :

$$1 + \tilde{\eta}\, u\ +\ \dots\ . \tag{6.8}$$

The solution to the boundary equation at infinity is:

$$\begin{aligned} f\ &=\ f_1\, r^{\Delta_+}\ +\ f_2\, r^{\Delta_-}, \\ \Delta_\pm &= \frac{-3 \pm \sqrt{9+16\alpha^2}}{2}. \end{aligned} \tag{6.9}$$

The first term is again not acceptable so the second term has to be chosen.

Finally we should find the appropriate set of parameters (temperature) for which the full equation admits a solution with the desired boundary conditions; and follow the procedure of the previous section, fixing the horizon condition and searching for parameters for which f vanishes at the boundary at infinity.

In figure (2) we show the plot of f at large r. We find that it does not vanish anywhere; therefore the equation does not admit a solution with the desired boundary condition at any temperature. Thus when $(\vec{\phi})^2 = const \neq 0$, the dyon solution, has no marginally stable mode, indicating stability, in accordance to its topological nature.

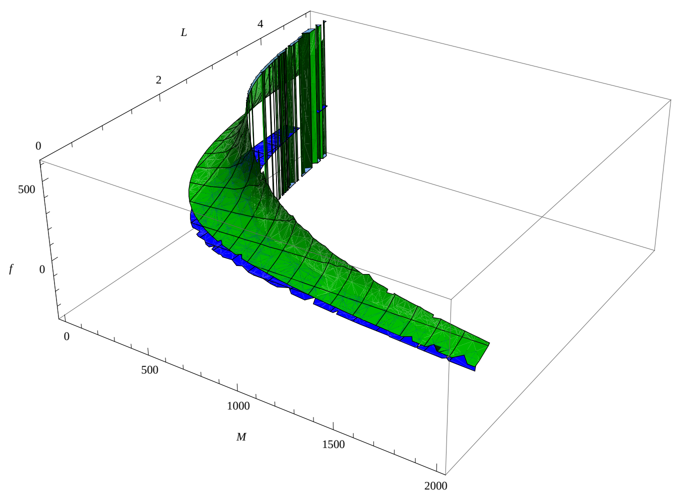


Figure 2: The green surface is the plot of f at very large r as a function of M and L; and the blue plane is where f vanishes. The plot is cut by $T \geq 0$ and $q^2 \geq 0$ conditions.

7 Conclusion

In this work we studied stability of two simple dyon solutions of SU(2) gauge theory in an asymptotically AdS blackhole background and found that the solution without the scalar field is unstable. In the context of holography this is a phase transition for boundary theory. The solution with the non-vanishing scalar, describing a topologically nontrivial dyon, turned

out to be stable as expected. The SU(2) color symmetry and the SU(2) space rotational symmetry of the model is broken to a diagonal SU(2) symmetry in both phases. However the phase transition has a topological character.

It is tempting to relate this holographic set up to a gravitational dual of a strong interaction physics model in 2+1 dimensions; the scalar iso-vector field of the model as pions which are believed to be Goldstone bosons of the spontaneously broken flavor symmetry. The color-spin locking in the dyon configuration is reminiscent of the color-flavor locking in QCD which is related to color superconductivity in neutron stars. And of course the intriguing possibility of associating the topological magnetic configuration with the phenomenon of confinement in QCD.

It would require further detailed study of the model to make any firm statements on these issues and we hope to get back to them.

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